PHYSICAL REVIEW E VOLUME 62, NUMBER 3 SEPTEMBER 2000

Analytic solution for two bubbles in a Hele-Shaw cell

Giovani L. Vasconcelos
Laboratório de Física Teórica e Computacional, Departamento de Física, Universidade Federal de Pernambuco, 50670-901, Recife, Brazil

(Received 22 February 2000; revised manuscript received 30 June 2000)

A four-parameter family of exact solutions is reported for two unequal bubbles moving steadily in a Hele-Shaw cell when surface tension is neglected. The solutions are given in closed form in terms of elliptic integrals and are found to be in good agreement with shapes observed in the experiments by Ikeda and Maxworthy [Phys. Rev. A **41**, 4367 (1990)]. An analytic solution for a finger with a bubble is also obtained as a special case of the general two-bubble solution. The relevance of these exact solutions for a selection theory for the cases of a finger with a bubble and a pair of bubbles is also briefly discussed.

PACS number(s): 47.55.Dz, 47.15.Hg, 47.54.+r

The displacement of a more viscous fluid by a less viscous one in a Hele-Shaw cell, where the fluids are confined between two closely spaced plates, has attracted a great deal of attention in the past two decades [1]. This research effort has been motivated in large part by the fact that the dynamics of interfaces in a Hele-Shaw cell exhibits mathematical analogies with other important moving-boundary problems, such as dendritic growth and directional solidification [2], while being more amenable to analytical and experimental investigation. In particular, the so-called *selection problem* for a finger in a Hele-Shaw channel has been extensively studied; this problem refers to the fact that, although there exists a whole family of solutions when surface tension is neglected, a unique finger is observed in the experiments. The selection problem for a finger dates back to the seminal paper by Saffman and Taylor [3] but only more recently it has been fairly well understood [4–8]. Experimental studies [9] have also been performed on a finger moving with a bubble attached at the tip, and here it was found that the finger became considerably narrower that the usual Saffman-Taylor finger. A similar effect has been observed for bubbles: when a small bubble attaches to the nose of a larger bubble, the larger bubble becomes more elongated and its velocity increases [10]. Although an approximate solvability theory has been proposed for the cases of a finger with a bubble at the tip [11] and a large bubble with a small one at the nose [12], the selection mechanisms in these two situations have not yet been completely elucidated.

In this Rapid Communication we report a four-parameter family of exact solutions for two unequal bubbles moving steadily in a Hele-Shaw cell when surface tension is neglected. Our solutions for the bubble shapes are written in closed form in terms of elliptic integrals and are found to be in excellent agreement with the shapes observed in the experiments by Ikeda and Maxworthy [10]. The solutions display the usual degeneracy of solutions with zero surface tension: fixing the bubble areas (and their relative separation) does not determine the bubble velocity. We also present an analytic solution for the case of a finger penetrating into a Hele-Shaw channel with a bubble moving ahead of the finger tip. This finger-bubble solution is obtained as a special case of our two-bubble solution in the limit that one of the bubble becomes infinitely elongated.

Before we present our solutions, it is perhaps worth mentioning that these analytic solutions at zero surface tension are of practical relevance in relation to the selection problem for a finger with a small bubble at the tip as well as for a large bubble with a small one at the nose. As already mentioned, a selection theory has been proposed for these two cases [11,12] that has as starting point the zero surface-tension solutions for a pure finger and a single bubble, respectively. This theory, however, is somewhat unsatisfactory in that the effect of the small bubble is taken into account through a rather unphysical condition, namely, that the interface has a cusp at the leading front (i.e., the finger tip and the large-bubble nose, respectively). In light of the exact solutions reported here, a more rigorous solvability analysis could in principle be carried out. Given that these solutions are expressed in terms of elliptic integrals, such an analysis is expected to be more mathematically challenging and will not be attempted in the present paper.

Here we consider the problem of two unequal bubbles moving with the same velocity U in a Hele-Shaw cell with the channel geometry. Without loss of generality, we assume that the viscous fluid has a unity velocity at infinity in front of and behind the bubbles. We suppose that the fluid inside the bubbles has negligible viscosity so that the pressure inside is constant. We will also neglect the interfacial tension, and hence, the fluid-pressure will be taken as constant along the bubble interface. We concern ourselves here with the case in which the bubbles are symmetrical about the channel centerline. For convenience, we will work in a reference frame in which the bubbles are stationary and choose our system of coordinates so that the origin is placed at the nose of the first bubble, with the channel walls being at $y=\pm 1$; see Fig. 1.

The exact solutions presented below are obtained via conformal mapping techniques. The idea here is to consider the conformal mapping $z=x+iy=f(\zeta)$ from the ζ complex plane onto the fluid region, with the two half-lines $\mathrm{Im}\,\zeta=0$, $|\zeta|\geqslant 1$ being mapped onto the two channel walls, respectively. In the ζ plane the bubbles correspond to two slits on the real axis along the intervals $-1<s_2\leqslant \mathrm{Re}\,\zeta\leqslant 0$ and $0<s_2\leqslant \mathrm{Re}\,\zeta\leqslant s_3<1$, for given s_1, s_2, and s_3. Using a variant of the method described in Ref. [13] one can construct explicitly the mapping function $f(\zeta)$ from which the inter-

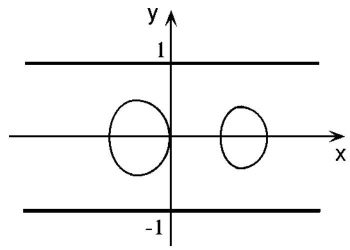

FIG. 1. Two bubbles in a Hele-Shaw cell.

faces are obtained in parametric form: $x(s)+iy(s)=f(s)$, where the two bubbles correspond to the intervals $s_1 \leqslant s \leqslant 0$ and $s_2 \leqslant s \leqslant s_3$, respectively. Here, owing to lack of space, we will not expound the conformal mapping formulation and will simply quote the final result for the bubble shapes. A complete derivation of our solution will be presented in a forthcoming publication [14].

In order to simplify our notation let us first define the following function:

$$H(\varphi,p,q,k)=G(\varphi,p,k)-G(\varphi,q,k), \tag{1}$$

where

$$G(\varphi,n,k)=\sqrt{(1-n)(1-k^2/n)}\left[\Pi(\varphi,n,k) - \frac{\Pi(n,k)}{K}F(\varphi,k)\right]. \tag{2}$$

Here $F(\varphi,k)$ and $\Pi(\varphi,n,k)$ are the normal elliptic integrals of first and third kinds, respectively, with $K \equiv K(k) \equiv F(\pi/2,k)$ and $\Pi(n,k) \equiv \Pi(\pi/2,n,k)$ being the complete elliptic integrals of first and third kinds, respectively. In the above we have adopted Legendre's notation for the elliptic integrals in which φ is the *argument*, k is called the *modulus*, and n is referred to as the *parameter* [15].

We are now in position to describe our two-bubble solution. The first bubble is given by the following parametric equations:

$$x(s)=\frac{2}{\pi}\frac{U-1}{U}\tanh^{-1}s, \tag{3}$$

$$y(s)=\pm\frac{2}{\pi}U^{-1}H(\varphi_1(s),p,q,k), \tag{4}$$

where

$$\varphi_1(s)=\sin^{-1}\sqrt{\frac{2s}{p-q+(p+q)s}}, \tag{5}$$

and the parameter s ranges over the interval $s_1 \leqslant s \leqslant 0$, with

$$s_1=\frac{q-p}{p+q-2}. \tag{6}$$

For the second bubble we have similar expressions:

$$x(s)=x_0+\frac{2}{\pi}\frac{U-1}{U}\tanh^{-1}s, \tag{7}$$

$$y(s)=\pm\frac{2}{\pi}U^{-1}H(\varphi_2(s),k^2/p,k^2/q,k), \tag{8}$$

where

$$\varphi_2(s)=\sin^{-1}\sqrt{\frac{p-q+(p+q)s}{2k^2s}}, \tag{9}$$

and the additional constant x_0 in Eq. (7) reads

$$x_0=\frac{U^{-1}}{K}[F(\beta_p,k')-F(\beta_q,k')], \tag{10}$$

with

$$\beta_p=\sin^{-1}\sqrt{\frac{1-p}{(k')^2}}, \tag{11}$$

and similarly for β_q. Here $k'=\sqrt{1-k^2}$ is the so-called *complementary modulus*. The parameter s for the second bubble takes values in the interval $s_2 \leqslant s \leqslant s_3$, where

$$s_2=\frac{q-p}{p+q}, \quad s_3=\frac{q-p}{p+q-2k^2}. \tag{12}$$

In the solution above, k, p, and q are free parameters satisfying the following condition:

$$0<k^2<p<q<1. \tag{13}$$

Besides the velocity U, the other relevant physical parameters are the areas of the two bubbles and the separation between them. Since the solutions above have four free parameters, namely, U, k, p, and q, it is clear that fixing the bubble sizes (and their relative separation) does not determine the bubble velocity. This is the usual degeneracy of solutions with zero surface tension [3,16]. Such degeneracy is expected to be removed by the effect of surface tension, as is the case for a finger (see, e.g., Ref. [2]) and a single bubble [17]. This issue, however, will not be addressed in the present paper.

For completeness, we wish to point out here that in the limit that the separation between the two bubbles vanishes (so that they merge into a single bubble) our two-bubble solution above correctly reproduces, as it must, the single bubble solution originally found by Taylor and Saffman [16]. This limit is achieved by letting the parameters k, p, and q all approach 1 but in such way that the quantities $m=(k^2-p)/(1-p)$ and $n=(k^2-q)/(1-q)$ remain finite and satisfy the relation $mn=1$. (This corresponds to taking $s_2=0$ and $s_3=-s_1$.) Using the special values for the elliptic integrals in these limits we can show that Eqs. (3) and (4) do indeed yield the solution given in Eq. (10) of Ref. [16]. We will not, however, present the mathematical details of such calculation here.

Comparison with experiments. Experiments on the motion of two bubbles in a Hele-Shaw cell have been performed by Ikeda and Maxworthy [10]. In their experiment a large bubble, injected at the bottom of an inclined Hele-Shaw cell, would rise to meet and attach to a small bubble that had been injected at a point further up the cell. The subsequent motion of the bubbles was recorded on video from which the bubble

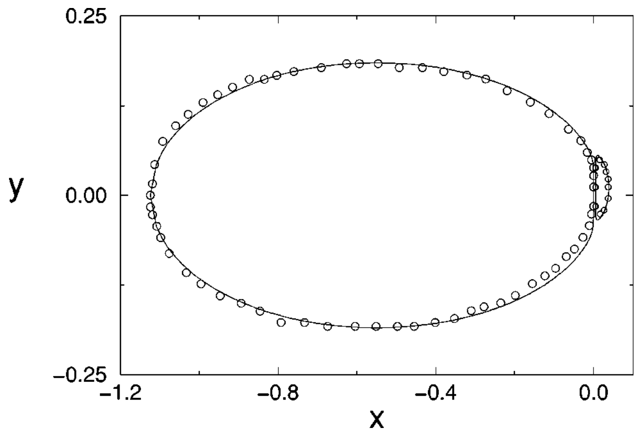

FIG. 2. A large bubble with a small bubble at the nose. (The bubbles are moving to the right.) The circles represent an observed shape taken from Fig. 1(a) of Ref. [10], while the solid line is an exact solution with $U'=3.4$, $s_1=-0.9857$, $s_2=10^{-5}$, and $s_3=0.0746$.

shapes and velocity were obtained. Before we present a comparison between our solutions and the observed shapes, we note that for the experimental setup used by Ikeda and Maxworthy [10] the velocity U appearing in the formulas above must be replaced with [3]

$$U'=1+\frac{U}{U^*}, \tag{14}$$

where $U^*=(b^2 g \sin\alpha/12\mu_1)(\rho_1-\rho_2)$. Here b is the cell gap, g is the gravitational acceleration, α is the tilting angle, ρ is the fluid density, μ is the fluid viscosity, and the subscripts 1 and 2 denote the fluids outside and inside the bubbles, respectively. In Fig. 2 we show one of the shapes reported in Ref. [10], to which we superimpose a member of the family of analytic solutions described above. As one can see in this figure, the exact solution describes remarkably well the observed shape for the large bubble, the only noticeable deviation occurring at the lower region behind the small bubble. This small discrepancy is probably due to the fact that the smaller bubble is slightly off the nose of the larger one, thus breaking slightly the centerline symmetry. Indeed, for the small bubble in Fig. 2 the best agreement between the theoretical curve and the experimental shape was obtained when the former was shifted slightly upward. We note furthermore that according to Eq. (14) our value $U'=3.4$ (see the caption of Fig. 2) corresponds to $U/U^*=2.4$, which is indeed very close to the value reported in Ref. [10]. [Rigorously speaking, our definition of U^* given above differs from that used by Ikeda and Maxworthy [10] by a correction factor $(D/L)_{U'=2}$. However, for the bubble shown in Fig. 2 such a factor is approximately equal to unity.]

Finger with a bubble at the tip. In the limit that the first bubble becomes infinitely elongated (while the second one remains of finite size), we arrive at a situation in which a finger penetrates into the channel with a bubble moving ahead of the finger tip. An explicit solution for this case can be obtained from our general solution above if we set $q=1$. (This corresponds to taking $s_1=-1$.) Using the fact that $\Pi(\varphi,1,k)$ remains finite for $0<k<1$ and $0<\varphi<\pi/2$ and that $\lim_{n\to 1}\Pi(n,k)=\pi/2k'\sqrt{1-n}$, one obtains that $G(\varphi,1,k)=-\pi F(\varphi,k)/2K$. In addition we have that $G(\varphi,k^2,k)\equiv 0$. From these results, it immediately follows that for $q=1$ Eqs. (4) and (8) respectively become

$$y_f(s)=\pm U^{-1}\left[\frac{2}{\pi}G(\varphi_f(s),p,k)+K^{-1}F(\varphi_f,k)\right], \tag{15}$$

$$y_b(s)=\pm\frac{2}{\pi}U^{-1}G(\varphi_b(s),k^2/p,k), \tag{16}$$

where

$$\varphi_f(s)=\sin^{-1}\sqrt{\frac{2s}{p-1+(p+1)s}}, \tag{17}$$

$$\varphi_b(s)=\sin^{-1}\sqrt{\frac{p-1+(p+1)s}{2k^2 s}}. \tag{18}$$

Here we have introduced the subscripts f and b to denote explicitly the solutions for the finger and the bubble, respectively. The parameter s for the finger and the bubble runs over the intervals $-1<s\leq 0$ and $s_2\leq s\leq s_3$, respectively, where now $s_2=(1-p)/(1+p)$, $s_3=(1-p)/(1+p-2k^2)$, and $0<k^2<p<1$. The coordinates $x(s)$ for the finger and the bubble remain as given in Eqs. (3) and (7), respectively, the only difference being that now the constant x_0 reads simply

$$x_0=\frac{F(\beta_p,k')}{UK}, \tag{19}$$

with β_p as in Eq. (11). We thus have a three-parameter family of analytic solutions for a finger with a bubble. (Exact solutions for a finger with bubbles have been reported previously by the author [13], but there the solutions were written in integral form and no representation in closed form was obtained.) As already mentioned, the analytic solutions for a finger with a bubble presented above are relevant for the experiments on narrow fingers performed by Couder *et al.* [9].

In conclusion, we have presented a four-parameter family of exact solutions for two unequal bubbles moving steadily in a Hele-Shaw cell when surface tension is neglected. Although these solutions are for the idealized case of zero surface tension, they nevertheless describe remarkably well shapes observed in actual experiments, particularly when the bubble size is much greater than the cell gap. An analytic solution for a finger with a bubble was also presented. The exact solutions reported in the present paper opens up the prospect for more detailed studies on the selection problem for a finger with a bubble as well as for a pair of bubbles. Here again it is expected that a small amount of surface will remove the degeneracy of the zero surface-tension solutions.

This work was supported in part by FINEP, CNPq, and PRONEX Project No. 76.97.1004.00.

[1] For reviews, see, e.g., D. Bensimon, L. Kadanoff, S. Liang, B. I. Shraiman, and C. Tang, Rev. Mod. Phys. **58**, 977 (1986); D. Kessler, J. Koplik, and H. Levine, Adv. Phys. **37**, 255 (1988).

[2] P. Pelcé, *Dynamics of Curved Fronts* (Academic, New York, 1988).

[3] P. G. Saffman and G. I. Taylor, Proc. R. Soc. London, Ser. A **245**, 312 (1958).

[4] B. I. Shraiman, Phys. Rev. Lett. **56**, 2028 (1986).

[5] D. C. Hong and J. S. Langer, Phys. Rev. Lett. **56**, 2032 (1986).

[6] R. Combescot, T. Dombre, V. Hakim, Y. Pomeau, and A. Pumir, Phys. Rev. Lett. **56**, 2036 (1986).

[7] S. Tanveer, Phys. Fluids **30**, 1589 (1987); in *Asymptotics Beyond All Orders*, edited by H. Segur *et al.* (Plenum Press, New York, 1991).

[8] M. B. Mineev-Weinstein, Phys. Rev. Lett. **80**, 2113 (1998).

[9] Y. Couder, N. Gérard, and M. Rabaud, Phys. Rev. A **34**, 5175 (1986); M. Rabaud, Y. Couder, and N. Gérard, *ibid.* **37**, 935 (1988).

[10] E. Ikeda and T. Maxworthy, Phys. Rev. A **41**, 4367 (1990).

[11] D. C. Hong and J. S. Langer, Phys. Rev. A **36**, 2325 (1987).

[12] D. C. Hong and F. Family, Phys. Rev. A **38**, 5253 (1988).

[13] G. L. Vasconcelos, Phys. Fluids **11**, 1281 (1999).

[14] G. L. Vasconcelos (unpublished).

[15] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals*, 2nd ed. (Springer, Berlin, 1971).

[16] G. I. Taylor and P. G. Saffman, Q. J. Mech. Appl. Math. **12**, 265 (1959).

[17] S. Tanveer, Phys. Fluids **29**, 3537 (1986).